

On Existence and Uniqueness of the Weak Solution of a Generalized Boussinesq Equation with Pressure and Mixed Boundary Conditions *

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Abstract

In this paper, the existence and uniqueness of weak solution for a generalized Boussinesq equation that couples the mass and heat flows in a viscous incompressible fluid is considered. The viscosity and the heat conductivity are assumed to depend on the temperature. The boundary condition on velocity of the fluid is non-standard where the dynamical pressure is given on some part of the boundary, and the temperature of the fluid is represented in a mixed boundary condition.

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1 Introduction

The existence and uniqueness of the solution for the generalized Boussinesq equation have been studied extensively by many authors, see [5, 6, 8, 9, 10] and the references therein. However, the boundary conditions in these works are Dirichlet type for velocity of the fluid. In [4] the blow-up and global existence for nonlinear parabolic equations with Neumann boundary conditions is considered. The recent works on the existence and uniqueness of the solution of Boussinesq equation

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can be found in [3, 7, 15, 18]. [18] studies the existence and uniqueness of stationary Boussinesq system with non-smooth mixed boundary conditions for the temperature, and non-smooth Dirichlet boundary condition for the velocity. The local existence and uniqueness of the solution for the boundary-value problem for the stationary Boussinesq heat and mass transfer equations with the inhomogeneous non-standard boundary conditions for the velocity, and the mixed boundary conditions for the temperature and concentration are obtained in [15]. [7] investigates the existence and uniqueness of weak solutions of the stationary Boussinesq system with the homogeneous Dirichlet boundary conditions for the velocity and temperature. The the existence and uniqueness of weak solutions of the time-periodic solutions for a generalized Boussinesq equation with Neumann boundary conditions for temperature are studied. The global regularity of the classical solutions for a 2D Boussinesq equation with the vertical viscosity and diffusivity has been investigated.

In this paper, we are concerned with the existence and uniqueness of the solution of a generalized Boussinesq equation with nonlinear diffusion of the velocity and temperature, where the press boundary condition on the velocity of the fluid and the mixed boundary condition on the temperature are given, The system is described by the following initial-boundary conditions:

$$\left\{ \begin{array}{ll} \frac{\partial z(x, t)}{\partial t} - \gamma(w(x, t)) \Delta z(x, t) + (z(x, t), \nabla) z(x, t) - \beta g w(x, t) \\ \quad \quad \quad = f_1(x, t) - \nabla \pi(x, t) & \text{on } Q, \\ \operatorname{div}(z(x, t)) = 0 & \text{on } Q, \\ \frac{\partial w(x, t)}{\partial t} - \operatorname{div}(k(w(x, t)) \nabla w(x, t)) + (z, \nabla) w = f_2(x, t) & \text{on } Q, \\ z(x, t)_\tau = 0, \pi(x, t) + \frac{1}{2} |z(x, t)|^2 = v_1(x, t), w(x, t) = 0 & \text{on } \Sigma_1, \\ z(x, t) = 0, -k(w(x, t)) \frac{\partial w(x, t)}{\partial n} = v_2(x, t) & \text{on } \Sigma_2, \\ z(x, 0) = z_1(x), w(x, 0) = w_0(x) & \text{on } \Omega, \end{array} \right. \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N (N = 2, 3)$ is a bounded domain with the smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, $Q = \Omega \times (0, T)$ for the given $T > 0$, $\Sigma_i = \Gamma_i \times (0, T), i = 1, 2$, $\Sigma = \Gamma \times (0, T)$, n is the normal vector exterior to Γ , $z(x, t) \in \mathbb{R}^N$ denotes the velocity of the fluid at $x \in \Omega$ and time $t \in (0, T)$, $\pi(x, t) \in \mathbb{R}$ is the hydrostatic pressure, $w(x, t) \in \mathbb{R}$ is the temperature, $f_1(x, t)$ is the external force, g is the gravitational vector function, $\gamma(\cdot) > 0$ is the kinematic viscosity, $k(\cdot) > 0$ is the thermal conductivity, β is a positive constant associated to the coefficient of volume expansion, $f_2(x, t)$ is the heat source strength, and v_1, v_2 are prescribed functions to be determined later. The i -th component of $(z(x, t), \nabla z(x, t))z(x, t)$ in Cartesian coordinate is given by

$$((z(x, t), \nabla) z(x, t))_i = \sum_{j=1}^N z_j(x, t) \frac{\partial z_j(x, t)}{\partial x_j}, \quad (z(x, t), \nabla) w(x, t) = \sum_{j=1}^N z_j(x, t) \frac{\partial w(x, t)}{\partial x_j},$$

and

$$z_\tau(x, t) = z(x, t) - z_n(x, t) \cdot n, \quad z_n(x, t) = (z(x, t) \cdot n).$$

The classical Boussinesq equation where γ and k are constants has been studied widely in literature, see for instance [11, 12]. Equation (1.1) that is paid less attention represents the physical system for which we the variation of the fluid viscosity, and the thermal conductivity with temperature cannot be ignored (see e.g., [10] and the references therein). Some researches on the generalized Boussinesq equation and the Boussinesq system with nonlinear thermal diffusion are available in [5, 6, 8, 9] but the boundary conditions in this works are homogeneous. It should be pointed out that the variation of the viscosity with the temperature is important in understanding the process of the flow. As a first step to this point, we establish, in this paper, the the existence of non stationary solution of Equation (1.1) where the dynamical pressure is given on some part of the boundary, and the boundary condition for temperature of the fluid is mixed. Due to the stronger nonlinear coupling between the equations, Equation (1.1) is much difficult than the classical Boussinesq equations.

We proceed as follows. In section 2, we give some preliminary results on Equation (1.1). With the Galerkin method, we show, in section 3 the existence of the global solution to the problem (1.1) under certain conditions on the temperature dependency of the viscosity and the thermal conductivity. The assumption on non-constant gravitational field would be helpful in some other geophysical models. The uniqueness of the weak solution is presented in section 4.

2 Preliminary results

For notation simplicity, we do not distinguish the functions defined on the real number field \mathbb{R} or the N -dimensional Euclidean space \mathbb{R}^N , which is clear from the context. The $L^2(\Omega)$ inner product and the norm induced by the inner product are denoted by (\cdot, \cdot) and $|\cdot|$ respectively. The norm of the Sobolev space $H^m(\Omega)$ is denoted by $\|\cdot\|_m$ which is reduced to $L^2(\Omega)$ when $m = 0$. $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. We use $\mathcal{D}(0, T)$ to denote the C^∞ -functions with the compact on $(0, T)$ and $\mathcal{D}(0, T)'$ the space of distribution associated with $\mathcal{D}(0, T)$.

Now we introduce some function spaces as follows.

$$\left\{ \begin{array}{l} D = \{\psi \in (C^\infty(\Omega))^N \mid \operatorname{div}(\psi(x)) = 0 \text{ on } \Omega \text{ and } \phi_\tau(x) = 0 \text{ on } \Gamma_2\}, \\ H = \text{completion of } D \text{ under the } (L^2(\Omega))^N \text{ - norm}, \\ V = \text{completion of } D \text{ under the } (H^1(\Omega))^N \text{ - norm}, \\ D_{\Gamma_1} = \{\varphi \in C^\infty(\Omega) \mid \varphi(x) = 0 \text{ on } \Gamma_1\}, \\ \tilde{H} = \text{closure of } D_{\Gamma_1} \text{ in } L^2(\Omega), \\ W = \text{closure of } D_{\Gamma_1} \text{ in } H^1(\Omega). \end{array} \right. \quad (2.1)$$

The norm of H is denoted by $|\cdot|$ and the norm of V is denoted by $\|\cdot\|$. If there is no the anxiety confused in the notation, then we will also denote the norm of \tilde{H} is by $|\cdot|$ and the norm of W is denoted by $\|\cdot\|$.

Suppose that (z, w) is classical solution of (1.1).

Now, specially, we assume that $v_1, f_1 \in L^2(0; T; (L^2(\Gamma_1))^N)$ and $v_1, f_2 \in L^2(0; T; (L^2(\Gamma_1)))$.

Multiplier the first equation of (1.1) by $\phi \in V$, integrate by parts over Ω and take the boundary condition into account to get

$$\frac{d}{dt}(z, \psi) + a_{\gamma(w)}(z, \psi) + b(z, z, \psi) - (\beta gw, \phi) = (f_1, \psi) + (\nu_1, \psi_n)_{\Gamma_1}, \quad \psi_n = (\psi \cdot n)n. \quad (2.2)$$

Multiplier the second equation of (1.1) by $\varphi \in W$, integrate by parts over Γ , and take the boundary condition into account to obtain

$$\frac{d}{dt}(w, \varphi) + a_{k(w)}(w, \varphi) + c(z, w, \varphi) = (f_2, \varphi) + (\nu_2, \varphi)_{\Gamma_2}. \quad (2.3)$$

Now let $\chi \in C^1[0, T]$ be a function such that $\chi(T) = 0$. Multiplier Equations (2.2) and (2.3) by χ respectively, and integrate by parts to yield

$$\begin{aligned} & \int_0^T [-(z(t), \phi \chi'(t)) + a_{\gamma(w)}(z(t), \phi \chi(t)) + b(z(t), z(t), \phi \chi(t)) \\ & + (\beta gw(t), \phi \chi(t))] dt = \int_0^T (f_1(t), \phi \chi(t)) dt \end{aligned} \quad (2.4)$$

$$\begin{aligned} & + \int_0^T \int_{\Gamma_1} (\nu_1, \phi_n \chi(t)) ds dt + (z_0, \phi) \chi(0). \\ & \int_0^T [-(w(t), \varphi \chi'(t)) + a_{k(w)}(w(t), \varphi \chi(t)) + c(z(t), w(t), \varphi \chi(t)) \\ &] dt = \int_0^T (f_2(t), \varphi \chi(t)) dt + \int_0^T \int_{\Gamma_2} (\nu_2, \varphi \chi(t)) ds dt + (w_0, \varphi) \chi(0), \end{aligned} \quad (2.5)$$

where we write $z(\cdot, t) = z(t)$, $w(\cdot, t) = w(t)$ by abuse of notation without confusion from the context, and

$$\begin{aligned} (z, \phi) &= \sum_{j=1}^N \int_{\Omega} z_j(x, \cdot) \phi_j(x) dx, \quad (w, \varphi) = \int_{\Omega} (w(x, \cdot) \varphi(x)) dx, \\ a_{\gamma(w)}(u, z) &= (\gamma(w) \text{rot} z(x, \cdot), \text{rot} \phi), \quad b(z, z, \phi) = \int_{\Omega} (\text{rot} z(x, \cdot) \times z(x, \cdot)) \phi(x) dx, \\ a_{k(w)}(w, \varphi) &= \sum_{j=1}^N \int_{\Omega} k(w) \frac{\partial w(x, \cdot)}{\partial x_j} \cdot \frac{\partial \varphi(x)}{\partial x_j} dx, \quad c(z, w, \varphi) = \sum_{j=1}^N \int_{\Omega} z_j(x, \cdot) \frac{\partial w(x, \cdot)}{\partial x_j} \varphi(x) dx. \end{aligned}$$

We are now in a position to state some preliminary results. Throughout the paper, we always assume that there are positive constants $\gamma_0, \gamma_1, k_0, k_1$ such that

$$\gamma_0 \leq \gamma(t) \leq \gamma_1, \quad k_0 \leq k(t) \leq k_1, \quad \forall t \in [0, \infty). \quad (2.6)$$

The following Lemmas 2.1-2.3 can be obtained by the Sobolev inequalities and the compactness theorem. We can also refer to theorem 1.1 of [14] on page 107 and lemmas 1.2, 1.3 of [14] on page 109 (see also chapter 2 of [13]). The similar arguments can also be found in lemmas 1, 5 of [2].

Lemma 2.1. *The bilinear forms $a_{\gamma(\cdot, \cdot)}$ and $a_{k(\cdot, \cdot)}$ are coercive over V and W respectively. That is, there exist constants $c_1, \acute{c}_1 > 0$ such that*

$$a_{\gamma(w)}(z, z) \geq \gamma_0 c_1 \|z\|_1^2, \quad \forall z \in V \quad \text{and} \quad a_{k(w)}(w, w) \geq k_0 \acute{c}_1 \|w\|_1^2, \quad \forall w \in W.$$

Lemma 2.2. *The trilinear form $b(\cdot, \cdot, \cdot)$ is a linear continuous functional with respect to each variable defined on $(H^1(\Omega))^N$. That is, there exist a constant $c_2 > 0$ such that*

$$|b(u, v, w)| \leq c_2 \|u\| \|v\| \|w\|, \forall u, v, w \in (H^1(\Omega))^N.$$

Moreover, the following properties hold true

- (i). $b(u, v, v) = 0, \forall u, v \in V$;
- (ii). $b(u, v, w) = -b(u, w, v), \forall u \in V, v, w \in (H^1(\Omega))^N$;
- (iii). If $u_m \rightarrow u$ weakly on V and $v_m \rightarrow v$ strongly on H , then $b(u_m, v_m, w) \rightarrow b(u, v, w), \forall w \in V, w \in V$.

Lemma 2.3. *The trilinear form $c(\cdot, \cdot, \cdot)$ is a linear continuous functional with respect to each variable defined on $V \times W \times W$. That is, there exist a constant $c_3 > 0$ such that*

$$|c(z, w, \varphi)| \leq c_3 \|z\| \|w\| \|\varphi\|, \forall z \in V, w, \varphi \in W.$$

Moreover, the following properties hold true

- (i). $c(u, w, w) = 0, \forall z \in V, w \in W$;
- (ii). $c(z, w, \varphi) = -c(z, \varphi, w), \forall z \in V, w, \varphi \in W$;
- (iii). If $z_m \rightarrow z$ weakly on V and $w_m \rightarrow w$ strongly on $L^2(\Omega)$, then $c(z_m, w_m, \varphi) \rightarrow c(z, w, \varphi), \forall z \in V, w \in L^2(\Omega), \varphi \in W$.

By considering (2.3), (2.4), we define the weak of (1.1) such as;

Definition 1. Let $Y \equiv Z \times W = (L^2(0, T; V) \cap L^\infty(0, T; H)) \times (L^2(0, T; W) \cap L^\infty(0, T; \tilde{H}))$. Suppose that

$$\begin{aligned} f_1 &\in L^2(0, T; V^*), f_2 \in L^2(0, T; W^*), v_1 \in L^2(0, T; (H^{-1/2}(\Gamma_1))^N), v_2 \in L^2(0, T; (H^{-1/2}(\Gamma_2))^N), \\ z_0 &\in H, w_0 \in L^2(\Omega), g \in L^\infty(\Omega). \end{aligned}$$

The pair $\{z, w\}$ is said to be a weak solution of (1.1) if it satisfies

$$\begin{cases} \{z, w\} \in Y, z' \in L^1(0, T; V^*), w' \in L^1(0, T; W^*), \\ (z', \phi) + a_{\gamma(w)}(z, \phi) + b(z, z, \phi) + (\beta w g, \phi) = \langle f_1, \phi \rangle + \langle v_1, \phi_n \rangle_{\Gamma_1}, \forall \phi \in V, \\ (w', \varphi) + a_{k(w)}(w, \varphi) + c(z, w, \varphi) = \langle f_2, \varphi \rangle + \langle v_2, \varphi \rangle_{\Gamma_2}, \forall \varphi \in W, \\ z(0) = z_0, w(0) = w_0, \end{cases} \quad (2.7)$$

where the inner product in $(L^2(\Omega))^N$ is also denoted by (\cdot, \cdot) without confusion from the context, and that in V^* and V (also $H^{-1}(\Omega)$ and $H_0^1(\Omega)$) are denoted by $\langle \cdot, \cdot \rangle$.

Next, we reformulate Equation (2.7) in to the operator equation. To this purpose, it is noticed that for a fixed $\phi \in V$, the functional $\phi \in V \rightarrow a_{\gamma(w)}(z, \phi)$ is linear continuous. So there exists an $A_\gamma z \in V^*$ such that

$$\langle A_\gamma z, \phi \rangle = a_{\gamma(w)}(z, \phi), \forall \phi \in V. \quad (2.8)$$

Similarly, for fixed $u, v \in V$, $w \in V \rightarrow b(u, v, w)$ is a linear continuous functional on V . Hence there exist a $B(u, v) \in V^*$ such that

$$\langle B(u, v), w \rangle = b(u, v, w), \forall w \in V. \quad (2.9)$$

We denote $B(u) = B(u, u)$. Define

$$L_1(\phi) = \langle f_1, \phi \rangle + \langle v_1, \phi_n \rangle_{\Gamma_1}, \forall \phi \in V. \quad (2.10)$$

Then L_1 a linear continuous functional on V and so there exists a constant $c_4 > 0$ such that

$$\|L_1\phi\| \leq c_4\|\phi\|_V, \forall \phi \in V. \quad (2.11)$$

Hence there exists a $F_1 \in V^*$ such that $L_1(\phi) = \langle F_1, \phi \rangle$ for all $\phi \in V$.

By the operators defined above, we can write the second equation of (2.7) as

$$\frac{dz}{dt} + A_\gamma z + B(z) + \beta gw = F_1. \quad (2.12)$$

Similarly, we have

$$\langle A_k w, \varphi \rangle = a_{k(w)}(w, \varphi), \langle C(z, w), \varphi \rangle = c(z, w, \varphi), A_k w, C(z, w) \in W^*, \forall \varphi \in W. \quad (2.13)$$

Define

$$L_2(\varphi) = \langle f_2, \varphi \rangle + \langle v_2, \varphi \rangle_{\Gamma_2}. \quad (2.14)$$

Then L_2 a linear continuous functional defined on W and so there exists a constant $c_5 > 0$ such that

$$\|L_2\varphi\| \leq c_5\|\varphi\|_W, \forall \varphi \in W. \quad (2.15)$$

Hence there exists a $F_2 \in W^*$ such that $L_2(\varphi) = \langle F_2, \varphi \rangle$ for all $\varphi \in W$. By these operators defined above, we can write the first equation of (2.7) as

$$\frac{dw}{dt} + A_k w + C(z, w) = F_2. \quad (2.16)$$

Combining (2.12) and (2.16), we can write (2.7) in the abstract evolution equation as follows:

$$\begin{cases} \frac{dz}{dt} + A_\gamma z + B(z) + \beta gw = F_1, \\ \frac{dw}{dt} + A_k w + C(z, w) = F_2, \\ z(0) = z_0, w(0) = w_0. \end{cases} \quad (2.17)$$

Lemma 2.4. *If $z \in L^2(0, T; V)$, then $B(z) \in L^1(0, T; V^*)$; and if $w \in L^2(0, T; W)$, then $C(z, w) \in L^1(0, T; W^*)$.*

Proof. Apply the Hölder inequality and the Sobolev embedding theorem $H^1(\Omega) \subset L^4(\Omega)$ to obtain

$$|(B(z), \phi)| = |b(z, z, \phi)| \leq c_6 \|z\|_1 \|z\|_{L^4} \|\phi\|_{L^4} \leq c_6 \|z\|_1^2 \|\phi\|_1$$

By the trace theorem from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$ again, there exists a constant $c_{11} > 0$ such that

$$\|w_{m_n}(t)\|_{H^{1/2}(\Gamma_2)} \leq c_{11}\|w_m(t)\|.$$

By substituting above inequality into (3.9), we obtain such as;

$$\frac{d}{dt}|w_m(t)|^2 + (k_0 - c_{11}\varepsilon^2)\|w_m(t)\|^2 \leq \frac{1}{k_0}\|f_2(t)\|_{W^*}^2 + \frac{1}{\varepsilon^2}\|v_2(t)\|_{H^{-1/2}(\Gamma_2)}^2. \quad (3.10)$$

Setting $\varepsilon^2 = k_0/(2c_{11})$ in (3.10) gives

$$\frac{d}{dt}|w_m(t)|^2 + \frac{k_0}{2}\|w_m(t)\|^2 \leq \frac{1}{k_0}\|f_2(t)\|_{W^*}^2 + \frac{2c_{11}}{k_0}\|v_2(t)\|_{H^{-1/2}(\Gamma_2)}^2. \quad (3.11)$$

Integrate (3.11) over $[0, T]$ with respect to t to give

$$\begin{aligned} |w_m(T)|^2 + \frac{k_0}{2} \int_0^T \|w_m(t)\|^2 dt &\leq \frac{1}{k_0} \int_0^T \|f_2(t)\|_{W^*}^2 dt + \frac{2c_{11}}{k_0} \int_0^T \|v_2(t)\|_{H^{-1/2}(\Gamma_2)}^2 dt \\ &+ |w_m(0)|^2 \leq \frac{1}{k_0} \int_0^T \|f_2(t)\|_{W^*}^2 dt + \frac{2c_{11}}{k_0} \int_0^T \|v_2(t)\|_{H^{-1/2}(\Gamma_2)}^2 dt + |w(0)|^2. \end{aligned} \quad (3.12)$$

Since the right-hand side of (3.12) is bounded, we have

$$\{w_m\} \text{ is a bounded sequence in } L^2(0, T; W). \quad (3.13)$$

Replace T by $t \in [0, T]$ in (3.12) to obtain

$$\text{ess sup}_t |w_m(t)|^2 \leq \frac{1}{k_0} \int_0^T \|f_2(t)\|_{W^*}^2 dt + \frac{2c_{11}}{\varepsilon^2} \int_0^T \|v_2(t)\|_{H^{-1/2}(\Gamma_2)}^2 dt + |w(0)|^2. \quad (3.14)$$

Hence

$$\{w_m\} \text{ is a bounded sequence in } L^\infty(0, T; L^2(\Omega)). \quad (3.15)$$

On the other hand, integrate (3.8) over $[0, T]$ with respect to t to give

$$\begin{aligned} |z_m(T)|^2 + \frac{\gamma_0}{2} \int_0^T \|z_m(t)\|^2 dt &\leq \frac{1}{\gamma_0} \int_0^T \|f_1(t)\|_{V^*}^2 dt \\ &+ \frac{2(\beta\|g\|_\infty + c_{10})}{\gamma_0} \beta\|g\|_\infty \int_0^T \|w_m(t)\|^2 dt \\ &+ \frac{2(\beta\|g\|_\infty + c_{10})}{\gamma_0} \int_0^T \|v_1(t)\|_{H^{-1/2}(\Gamma_1)}^2 dt + |z_0|^2. \end{aligned} \quad (3.16)$$

Therefore

$$\{z_m\} \text{ is a bounded sequence in } L^2(0, T; V). \quad (3.17)$$

Replace T by $t \in [0, T]$ in (3.16) to get

$$\begin{aligned} \text{ess sup}_t |z_m(t)|^2 &\leq \frac{1}{\gamma_0} \int_0^T \|f_1(t)\|_{V^*}^2 dt + \frac{2(\beta\|g\|_\infty + c_{10})}{\gamma_0} \beta\|g\|_\infty \int_0^T \|w_m(t)\|^2 dt \\ &+ \frac{2(\beta\|g\|_\infty + c_{10})}{\gamma_0} \int_0^T \|v_1(t)\|_{H^{-1/2}(\Gamma_1)}^2 dt + |z_0|^2. \end{aligned} \quad (3.18)$$

Therefore,

$$\{z_m\} \text{ is a bounded sequence in } L^\infty(0, T; H). \quad (3.19)$$

(3.3) and (3.4) then follow from (3.13), (3.17), (3.15) and (3.19). \square

Lemma 3.2. *Let $\{z_m, w_m\}$ be the sequence determined by Lemma 3.1. Then there exists a sequence of $\{z_m, w_m\}$, still denoted by itself without confusion, such that*

$$z_m \rightarrow z \text{ strongly in } L^2(0, T; H), \quad w_m \rightarrow w \text{ strongly in } L^2(0, T; \tilde{H}). \quad (3.20)$$

Proof. By virtue of Lemmas 2.1-2.3, we can write (3.2) as follows:

$$\begin{cases} \left\langle \frac{dz_m(t)}{dt}, u_j \right\rangle = \langle F_1 - \beta g w_m(t) - A_\gamma z_m(t) - B(z_m(t), u_j), \forall j = 1, 2, \dots, m, \\ \left\langle \frac{dw_m(t)}{dt}, \mu_j \right\rangle = \langle F_2 - C(z_m(t), w_m(t)) - A_k w_m(t), \mu_j \rangle, \forall j = 1, 2, \dots, m. \end{cases} \quad (3.21)$$

Denote by $\{\tilde{z}_m, \tilde{w}_m\}$ the $\{z_m, w_m\}$ with zero values outside of $[0, T]$ and $\{\hat{z}_m, \hat{w}_m\}$ the Fourier transformations of $\{\tilde{z}_m, \tilde{w}_m\}$. We claim that there exists a $\nu > 0$ such that

$$\int_{-\infty}^{\infty} |\tau|^{2\nu} \|\hat{z}_m(\tau)\|^2 d\tau < \infty. \quad (3.22)$$

To this end, we write the first equation of (3.21) as

$$\frac{d}{dt}(\tilde{z}_m, u_j) = \langle \tilde{f}_{1m}, u_j \rangle + (z_{0m}, u_j)\delta_0 - (z_m(T), u_j)\delta_T, \quad (3.23)$$

where δ_0, δ_T are Dirac functions, and

$$\begin{aligned} \tilde{f}_{1m}(t) &= f_{1m}(t) \text{ for } t \in [0, T] \text{ and } \tilde{f}_{1m}(t) \text{ for } t > T, \\ f_{1m}(t) &= F_1 - \beta g w_m(t) - A_\gamma z_m(t) - B(z_m(t)). \end{aligned}$$

Take Fourier transform for Equation (3.23) to get

$$2\pi i \tau (\hat{z}_m, u_j) = \langle \hat{f}_{1m}, u_j \rangle + (z_{0m}, u_j) - (z_m(T), u_j) e^{-2\pi i T \tau}, \quad (3.24)$$

where \hat{f}_{1m} is the Fourier transform of \tilde{f}_{1m} .

Let $\tilde{q}_{jm}(t)$ be the function of q_{im} in $z_m(t) = \sum_{j=1}^m q_{jm}(t) u_j$ that is zero outside of $[0, T]$ and let $\hat{q}_{jm}(\tau)$ be its Fourier transform. Multiplier Equation (3.24) by \hat{q}_{jm} and sum for j from 1 to m to obtain

$$2\pi i |\hat{z}_m(\tau)|^2 = \langle \hat{f}_{1m}(\tau), \hat{z}_m(\tau) \rangle + (z_{0m}(0), \hat{z}_m(\tau)) - (z_m(T), \hat{z}_m(\tau)) e^{-2\pi i T \tau}. \quad (3.25)$$

We thus conclude that

$$\int_0^T \|f_{1m}(t)\|_{V^*}^2 dt \leq \int_0^T [\|f_1(t)\|_{V^*} + c_2 \|w_m(t)\| + \gamma_1 \|z_m(t)\| + c_1 \|z_m(t)\|^2] dt, \quad (3.26)$$

where we used the fact that $\|Bz_m(t)\| \leq c_1 \|z_m(t)\|^2$. By (3.13), (3.15) and (3.17), it follows from (3.26) that

$$\sup_{\tau \in \mathbb{R}} \|\hat{f}_{1m}(\tau)\|_{V^*} < \infty. \quad (3.27)$$

Apply (3.27) and the facts $\sup_{m \in \mathbb{Z}^+} [\|z_m(0)\| + \|z_m(T)\|] < \infty$ to (3.25) to yield

$$|\tau| |\hat{z}_m(\tau)|^2 \leq \acute{c}_3 \|\hat{z}_m(\tau)\| + \acute{c}_4 |\hat{z}_m(\tau)| = \acute{c}_5 \|\hat{z}_m(\tau)\|, \quad \acute{c}_5 = \acute{c}_3 + \acute{c}_4. \quad (3.28)$$

For ν , fixed, $\nu < 1/4$, we can observe that

$$|\tau|^{2\nu} \leq c'(\nu) \frac{1+|\tau|}{1+|\tau|^{1-2\nu}}, \forall \tau \in \mathbb{R}$$

From this inequality, we obtain

$$\int_{-\infty}^{\infty} |\tau|^{2\nu} |\hat{z}_m(\tau)|^2 d\tau \leq c'(\nu) \int_{-\infty}^{\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\nu}} |\hat{z}_m(\tau)|^2 d\tau \quad (3.29)$$

Thus by (3.28), we can obtain the constants $c'_0 > 0$ and $c'_1 > 0$ such that

$$\begin{aligned} \int_{-\infty}^{\infty} |\tau|^{2\nu} |\hat{z}_m(\tau)|^2 d\tau &\leq c'(\nu) \int_{-\infty}^{\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\nu}} |\hat{z}_m(\tau)|^2 d\tau \leq \\ &\leq c'_0 \int_{-\infty}^{\infty} \frac{\|\hat{z}_m(\tau)\|}{1+|\tau|^{1-2\nu}} d\tau + c'_1 \int_{-\infty}^{\infty} \|\hat{z}_m(\tau)\|^2 d\tau \end{aligned}$$

By (3.20) and the Parseval equality the last integral is bounded as $m \rightarrow \infty$; thus (3.22) will be proved if we show that;

$$\int_{-\infty}^{\infty} \frac{\|\hat{z}_m(\tau)\|}{1+|\tau|^{1-2\nu}} d\tau \leq Const$$

By the Schwarz inequality and the Parseval equality we can obtain

$$\int_{-\infty}^{\infty} \frac{\|\hat{z}_m(\tau)\|}{1+|\tau|^{1-2\nu}} d\tau \leq \left(\int_{-\infty}^{\infty} \frac{d\tau}{(1+|\tau|^{1-2\nu})^2} \right)^{1/2} \left(\int_0^T \|\hat{z}_m(t)\|^2 dt \right)^{1/2}$$

which is finite since $\nu < 1/4$, and bounded as $m \rightarrow \infty$; by (3.19). The proof of (3.22) is achieved.

By (3.22), (3.17) and (3.19), we conclude that

$$\{z_m\} \text{ is bounded in } H^\nu(\mathbb{R}; V) \cap H^\nu(\mathbb{R}; H). \quad (3.30)$$

By Lemma 2.5, there exists a subsequence of $\{z_m, w_m\}$ that is still denoted by itself without confusion such that

$$z_m \rightarrow z \text{ strongly in } L^2(0, T; H), \quad w_m \rightarrow w \text{ strongly in } L^2(0, T; L^2(\Omega)).$$

This is (3.20). □

Theorem 3.1. *Suppose that the functions γ, k satisfy that when $w_m \rightarrow w$ in $L^2(0, T; L^2(\Omega))$, then $\gamma(w_m) \rightarrow \gamma(w)$, $k(w_m) \rightarrow k(w)$ in $L^2(0, T; L^2(\Omega))$, respectively. Then there exists a weak solution to (1.1).*

Proof. Let Ψ and θ be continuous differentiable vector functions defined on $[0, T]$ with $\Psi(T) = \theta(T) = 0$. Multiply the first equation of (3.2) by Ψ and integrate over $[0, T]$ with respect to t to give

$$\begin{aligned} & - \int_0^T (z_m(t), \Psi'(t)u_j) dt + \int_0^T [a_{\gamma(w_m)}(z_m(t), \Psi(t)u_j) + b(z_m(t), z_m(t), \Psi(t)u_j) \\ & + \beta(w_m(t)g, \Psi(t)u_j)] dt = (z_{0m}, u_j)\Psi(0) + \int_0^T \langle f_1(t), u_j \rangle dt + \int_0^T \langle v_1, \Psi(t)u_{j_n} \rangle_{\Gamma_1} dt. \end{aligned} \quad (3.31)$$

Multiply the second equation of (3.2) by θ and integrate over $[0, T]$ with respect to t to give

$$\begin{aligned} & - \int_0^T (w_m(t), \theta'(t)\mu_j)dt + \int_0^T [a_{k(w_m)}(w_m(t), \theta(t)\mu_j) + c(z_m(t), w_m(t), \theta(t)\mu_j) \\ &]dt = (w_{0m}, \mu_j)\theta(0) + \int_0^T \langle f_2(t), \theta(t)\mu_j \rangle dt + \int_0^T \langle v_2, \theta(t)\mu_j \rangle_{\Gamma_2} dt \end{aligned} \quad (3.32)$$

Passing to the limit as $m \rightarrow \infty$ in (3.31) and (3.32) by applying (3.3), (3.20), the properties (iii) in Lemmas 2.2 and 2.3 for b and c , and the continuous assumption on γ and k , we obtain

$$\begin{aligned} & - \int_0^T (z(t), \Psi'(t)u_j)dt + \int_0^T [a_{\gamma(w)}(z(t), \Psi(t)u_j) + b(z(t), z(t), \Psi(t)u_j) \\ & + \beta(w(t)g, \Psi(t)u_j)]dt = (z_0, u_j)\Psi(0) + \int_0^T \langle f_1(t), u_j \rangle dt + \int_0^T \langle v_1, \Psi(t)u_{j_n} \rangle_{\Gamma_1} dt. \end{aligned} \quad (3.33)$$

$$\begin{aligned} & - \int_0^T (w(t), \theta'(t)\mu_j)dt + \int_0^T [a_{k(w)}(w(t), \theta(t)\mu_j) + c(z(t), w(t), \theta(t)\mu_j)]dt \\ & = (w_0, \mu_j)\theta(0) + \int_0^T \langle f_2(t), \theta(t)\mu_j \rangle dt + \int_0^T \langle v_2, \theta(t)\mu_j \rangle_{\Gamma_2} dt, \end{aligned} \quad (3.34)$$

where in obtaining (3.33) and (3.34), we used the following facts:

- The convergence of the nonlinear terms in $b(\cdot, \cdot, \cdot)$ and $c(\cdot, \cdot, \cdot)$ can be obtained in the same way as that in Chapter 3 of [13].

•

$$\begin{aligned} \int_0^T a_{\gamma(w_m)}(z_m(t), \Psi(t)\mu_j)dt &= \int_0^T (\gamma(w_m)\text{rot}z_m(t), \Psi(t)\text{rot}u_j)dt \\ &= \int_0^T (\text{rot}z_m(t), \gamma(w_m)\text{rot}\Psi(t)\text{rot}u_j)dt \\ &\rightarrow \int_0^T (\text{rot}z(t), \gamma(w)\Psi(t)\text{rot}u_j)dt = \int_0^T (\gamma(w)\text{rot}z(t), \Psi(t)\text{rot}u_j)dt \\ &= \int_0^T a_{\gamma(w)}(z(t), \Psi(t)\mu_j)dt, \end{aligned}$$

where we used the facts that $\text{rot}z_m \rightarrow \text{rot}z$ weakly in $L^2(0, T; V)$ and $\gamma(w_m)\Psi(t)\text{rot}u_j \rightarrow \gamma(w)\Psi(t)\nabla u_j$ strongly in $L^2(0, T; H)$ from the assumptions of Theorem 3.1.

- Similarly

$$\int_0^T a_{k(w_m)}(w_m(t), \theta(t)\mu_j)dt \rightarrow \int_0^T a_{\gamma(w)}(w(t), \theta(t)\mu_j)dt$$

by the facts again $\nabla w_m \rightarrow \nabla w$ weakly in $L^2(0, T; W)$ and $k(w_m)\theta(t)\nabla\mu_j \rightarrow k(w)\theta(t)\nabla\mu_j$ strongly in $L^2(0, T; \tilde{H})$ from the assumptions of Theorem 3.1.

By the density arguments, we have that (3.33) and (3.34) hold true for any $\phi \in V$ instead of u_j and $\varphi \in W$ instead on μ_j , respectively. That is,

$$\begin{aligned} & - \int_0^T (z(t), \Psi'(t)\phi)dt + \int_0^T [a_{\gamma(w)}(z(t), \Psi(t)\phi) + b(z(t), z(t), \Psi(t)\phi) \\ & + \beta(w(t)g, \Psi(t)\phi)]dt = (z_0, \phi)\Psi(0) + \int_0^T \langle f_1(t), \phi \rangle dt + \int_0^T \langle v_1, \Psi(t)\phi_n \rangle_{\Gamma_1} dt, \forall \phi \in V; \end{aligned} \quad (3.35)$$

$$\begin{aligned}
& - \int_0^T (w(t), \theta'(t)\varphi)dt + \int_0^T [a_{k(w)}(w(t), \theta(t)\varphi) + c(z(t), w(t), \theta(t)\varphi)]dt \\
& = (w_0, \varphi)\theta(0) + \int_0^T \langle f_2(t), \theta(t)\varphi \rangle dt + \int_0^T \langle v_2, \theta(t)\varphi \rangle_{\Gamma_2} dt, \forall \varphi \in W.
\end{aligned} \tag{3.36}$$

Now take $\Psi \in (D(0, T))^N$ in (3.35) and $\theta \in D(0, T)$ in (3.36). Then $\{z, w\}$ satisfies

$$\begin{cases} (z', \phi) + a_{\gamma(w)}(z, \phi) + b(z, z, \phi) + (\beta wg, \phi) = \langle f_1, \phi \rangle + \langle v_1, \phi_n \rangle_{\Gamma_1}, \forall \phi \in V, \\ (w', \varphi) + a_{k(w)}(w, \varphi) + c(z, w, \varphi) = \langle f_2, \varphi \rangle + \langle v_2, \varphi \rangle_{\Gamma_2}, \forall \varphi \in W. \end{cases} \tag{3.37}$$

This is the equations in (2.7). Finally, we determine the initial value of $\{z, w\}$. Actually, multiply the first equation of (3.37) and integrate over $[0, T]$ with respect to t to get

$$\begin{aligned}
& - \int_0^T (z(t), \Psi'(t)\phi)dt + \int_0^T [a_{\gamma(w)}(z(t), \Psi(t)\phi) + b(z(t), z(t), \Psi(t)\phi) \\
& + \beta(w(t)g, \Psi(t)\phi)]dt = (z(0), \phi)\Psi(0) + \int_0^T \langle f_1(t), \phi \rangle dt + \int_0^T \langle v_1, \Psi(t)\phi_n \rangle_{\Gamma_1} dt, \forall \phi \in V.
\end{aligned} \tag{3.38}$$

Subtract (3.38) from (3.35) to get $(z(0) - z_0, \phi)\Psi(0) = 0$. Take Ψ so that $\Psi(0) = 1$ we get $(z(0) - z_0, \phi) = 0$ for all $\phi \in V$. So $z(0) = z_0$. The similar arguments lead to $w(0) = w_0$. The proof is complete. \square

4 Uniqueness of the weak solution

Theorem 4.1. *Let $\gamma, k : L^2(\Omega) \rightarrow L^\infty(\Omega)$ are Lipschitz continuous functions, that is, there are constants $h_1, h_2 > 0$ such that*

$$\begin{aligned}
& \|\gamma(w_*) - \gamma(w_{**})\|_{L^\infty} \leq l_1 \|w_* - w_{**}\|, t \in [0, T] \text{ a.e., } \forall w_*, w_{**} \in L^2(\Omega), \\
& \|k(w_*) - k(w_{**})\|_{L^\infty} \leq l_2 \|w_* - w_{**}\|, t \in [0, T] \text{ a.e., } \forall w_*, w_{**} \in L^2(\Omega).
\end{aligned} \tag{4.1}$$

Suppose that the weak solution $\{z, w\}$ of (1.1) claimed by Theorem 3.1 satisfies

$$4d \left(\frac{\|z(t)\|_{L^4}}{c_1} + \frac{d}{k_0 c_1 \acute{c}_1} \|w(t)\|_{L^4}^2 \right) < \gamma_0, \forall t \in [0, T] \text{ a.e.,} \tag{4.2}$$

where $d > 0$ is the constant in Sobolev inequality that $\|f\|_{L^4(\Omega)} \leq d\|f\|_{H^1(\Omega)}$ for all $f \in H^1(\Omega)$, which depends on N and Ω , and the constants c_1 and \acute{c}_1 are that in Lemma 2.1. Then the weak solution is unique.

Proof. Suppose that we have two weak solutions $\{z_*, w_*\}, \{z_{**}, w_{**}\}$ to (1.1). Set $z = z_* - z_{**}, w = w_* - w_{**}$. Then by (2.7)

$$\begin{cases} \frac{d}{dt}(z, \phi) + (\gamma(w_*)\nabla z, \nabla \phi) + b(z_*, z_*, \phi) - b(z_{**}, z_{**}, \phi) + (\beta wg, \phi) \\ \quad = ((\gamma(w_*) - \gamma(w_{**}))\nabla z_{**}, \nabla \phi), \forall \phi \in V, \\ \frac{d}{dt}(w, \varphi) + (k(w_*)\nabla w, \nabla \varphi) + c(z_*, w_*, \varphi) - c(z_{**}, w_{**}, \varphi) \\ \quad = -((k(w_*) - k(w_{**}))\nabla w_{**}, \nabla \varphi), \forall \varphi \in W. \end{cases} \tag{4.3}$$

Set $\phi = z, \varphi = w$ in (4.3) to get

$$\left\{ \begin{array}{l} \frac{d}{dt}(z, z) + (\gamma(w_*)\nabla z, \nabla z) + b(z_*, z_*, z) - b(z_{**}, z_{**}, z) + (\beta w g, z) \\ \quad = ((\gamma(w_*) - \gamma(w_{**}))\nabla z_{**}, \nabla \phi), \\ \frac{d}{dt}(w, w) + (k(w_*)\nabla w, \nabla w) + c(z_*, w_*, w) - c(z_{**}, w_{**}, w) \\ \quad = -((k(w_*) - k(w_{**}))\nabla w_{**}, \nabla w). \end{array} \right. \quad (4.4)$$

This together with $b(z_{**}, z, z) = c(z_{**}, w, w) = 0$ gives

$$\left\{ \begin{array}{l} \frac{d}{dt}\|z(t)\|^2 + 2\gamma_0\|\nabla z(t)\|^2 \leq 2b(z, z, z_*) + 2(\beta g w, z) - 2((\gamma(w_*) - \gamma(w_{**}))\nabla z_{**}, \nabla z), \\ \frac{d}{dt}\|w(t)\|^2 + 2k_0\|\nabla w(t)\|^2 \leq 2c(z, w, w_*) + 2((k(w_*) - k(w_{**}))\nabla w_{**}, \nabla w). \end{array} \right. \quad (4.5)$$

By the Lipschitz continuity (4.1), for any given $\varepsilon > 0$, it follows from the “ z part” of (4.5) that

$$\left\{ \begin{array}{l} \frac{d}{dt}\|z(t)\|^2 + 2\gamma_0\|\nabla z(t)\|^2 \leq 2\|z\|_{L^4}\|\nabla z\|\|z_*\|_{L^4} + 2\beta\|g\|_\infty\|w\|\|z\| + 2\|\gamma(w_*) \\ \quad - \gamma(w_{**})\|_{L^\infty}\|\nabla z_{**}\|\|\nabla z\| \\ \leq 2\|z\|_{L^4}\|\nabla z\|\|z_*\|_{L^4} + 2\beta\|g\|_\infty\|w\|\|z\| + 2l_1\|w_* - w_{**}\|_{L^2}\|\nabla z_{**}\|\|\nabla z\| \\ \leq 2\|z\|_{L^4}\|\nabla z\|\|z_*\|_{L^4} + 2\beta\|g\|_\infty\|w\|\|z\| + 2l_1\left(\frac{\gamma_0}{2\varepsilon^2 c_1}\|w\|_{L^2}^2\|\nabla z_{**}\|^2 + \frac{\varepsilon^2 c_1}{\gamma_0}\|\nabla z\|^2\right). \end{array} \right. \quad (4.6)$$

Putting $\varepsilon^2 = \gamma_0^2/l_1$ in (4.6), we obtain

$$\frac{d}{dt}\|z(t)\|^2 + 2\gamma_0\|\nabla z(t)\|^2 \leq 2\|z\|_{L^4}\|\nabla z\|\|z_*\|_{L^4} + 2\beta\|g\|_\infty\|w\|\|z\| + \frac{l_1}{2\gamma_0 c_1}\|w\|_{L^2}^2\|\nabla z_{**}\|. \quad (4.7)$$

Similar arguments to the “ w part” of (4.5), we have

$$\frac{d}{dt}\|w(t)\|^2 + 2k_0\|\nabla w(t)\|^2 \leq 2\|z\|_{L^4}\|\nabla w\|\|w_*\|_{L^4} + \frac{l_2}{2k_0 c'_1}\|w\|_{L^2}^2\|\nabla w_{**}\|^2. \quad (4.8)$$

Sum (4.7) and (4.8) to get

$$\begin{aligned} & \frac{d}{dt}[\|z(t)\|^2 + \|w(t)\|^2] + \gamma_0\|\nabla z(t)\|^2 + k_0\|\nabla w(t)\|^2 \\ & \leq 2\|z\|_{L^4}\|\nabla z\|\|z_*\|_{L^4} + 2\beta\|g\|_\infty\|w\|\|z\| + \frac{l_1}{2\gamma_0 c_1}\|w\|_{L^2}^2\|\nabla z_{**}\| + 2\|z\|_{L^4}\|\nabla w\|\|w_*\|_{L^4} \\ & \quad + \frac{l_2}{2k_0 c'_1}\|w\|_{L^2}^2\|\nabla w_{**}\|^2. \end{aligned} \quad (4.9)$$

Apply the Hölder inequality and the Sobolev inequality to the right-hand side of (4.9) to obtain,

for any given $\varepsilon_1 > 0$, that

$$\begin{aligned}
& \frac{d}{dt} [\|z(t)\|^2 + \|w(t)\|^2] + \gamma_0 \|\nabla z(t)\|^2 + k_0 \|\nabla w(t)\|^2 \\
& \leq 2d \|\nabla z\|^2 \|z_*\|_{L^4} + 2\beta \|g\|_\infty \|w\| \|z\| + \frac{l_1}{2\gamma_0 c_1} \|w\|_{L^2}^2 \|\nabla z_{**}\| + 2d \|\nabla z\| \|\nabla w\| \|w_*\|_{L^4} \\
& \quad + \frac{l_2}{2k_0 \acute{c}_1} \|w\|_{L^2}^2 \|\nabla w_{**}\|^2 \\
& \leq 2d \|\nabla z\|^2 \|z_*\|_{L^4} + 2\beta \|g\|_\infty \|w\| \|z\| + \frac{l_1}{2\gamma_0 c_1} \|w\|_{L^2}^2 \|\nabla z_{**}\| \\
& \quad + 2d \left(\frac{\varepsilon_0^2}{2} \|\nabla w\|^2 + \frac{1}{\varepsilon_1^2} \|\nabla z\|^2 \|w_*\|_{L^4}^2 \right) + \frac{l_2}{2k_0 \acute{c}_1} \|w\|_{L^2}^2 \|\nabla w_{**}\|^2.
\end{aligned} \tag{4.10}$$

where $d > 0$ is the constant from Sobolev inequality that $\|f\|_{L^4(\Omega)} \leq d \|f\|_{H^1(\Omega)}$ for all $f \in H^1(\Omega)$.

Set $\varepsilon_1^2 = k_0 \acute{c}_1 / 2d$ in (4.10) to get

$$\begin{aligned}
& \frac{d}{dt} [|z(t)|^2 + |w(t)|^2] + \gamma_0 c_1 \|z(t)\|^2 + \frac{k_0 \acute{c}_1}{2} \|w(t)\|^2 \\
& \leq 2d \|z\|^2 \|z_*\|_{L^4} + \beta \|g\|_\infty (|w|^2 + |z|^2) + \frac{l_1}{2\gamma_0 c_1} |w|^2 \|z_{**}\| + \frac{2d^2}{k_0 \acute{c}_1} \|z\|^2 \|w_*\|_{L^4}^2 \\
& \quad + \frac{l_2}{2k_0 \acute{c}_1} |w|^2 \|w_{**}\|^2 \\
& \leq \left(2d \|z_*\|_{L^4} + \frac{2d^2}{k_0 \acute{c}_1} \|w_*\|_{L^4}^2 \right) \|z\|^2 + \left(\beta \|g\|_\infty + \frac{l_1}{2\gamma_0^2 c_1} \|z_{**}\|^2 + \frac{l_2}{2k_0 \acute{c}_1} \|w_{**}\|^2 \right) |w|^2 \\
& \quad + \beta \|g\|_\infty |z|^2 \\
& \leq \frac{\gamma_0 c_1}{2} \|z\|^2 + \left(\beta \|g\|_\infty + \frac{l_1}{\gamma_0 c_1} \|z_{**}\|^2 + \frac{l_2}{2k_0 \acute{c}_1} \|w_{**}\|^2 \right) |w|^2 + \beta \|g\|_\infty |z|^2.
\end{aligned} \tag{4.11}$$

In the first term of the last row of (4.11), we used the assumption (4.2). From (4.11), we conclude that

$$\begin{aligned}
& \frac{d}{dt} [|z(t)|^2 + |w(t)|^2] + \frac{\gamma_0 c_1}{2} \|z(t)\|^2 + \frac{k_0 \acute{c}_1}{2} \|w(t)\|^2 \\
& \leq \left(\beta \|g\|_\infty + \frac{l_1}{2\gamma_0 c_1} \|z_{**}\|^2 + \frac{l_2}{2k_0 \acute{c}_1} \|w_{**}\|^2 \right) |w|^2 + \beta \|g\|_\infty |z|^2,
\end{aligned}$$

from which we obtain

$$\frac{d}{dt} [|z(t)|^2 + |w(t)|^2] \leq M(t) |w(t)|^2 + N |z(t)|^2 \tag{4.12}$$

or

$$\frac{d}{dt} [|z(t)|^2 + |w(t)|^2] \leq (M(t) + N) [|w(t)|^2 + |z(t)|^2], \tag{4.13}$$

where

$$M(t) = \beta \|g\|_\infty + \frac{l_1}{2\gamma_0 c_1} \|z_{**}(t)\|^2 + \frac{l_2}{2k_0 \acute{c}_1} \|w_{**}(t)\|^2, N = \beta \|g\|_\infty.$$

Since $M(\cdot) + N$ is integrable in $[0, T]$ with respect to t , we obtain, from (4.13), that

$$\frac{d}{dt} \left\{ e^{-(M+N)} [|z(t)|^2 + |w(t)|^2] \right\} \leq 0, \forall t \in [0, T] \text{ a.e..} \tag{4.14}$$

This together with $z(0) = w(0) = 0$ gives

$$|z(t)|^2 + |w(t)|^2 \leq 0, \forall t \in [0, T] \text{ a.e..}$$

Therefore, $z_* = z_{**}, w_* = w_{**}$. The proof is complete. \square

Remark 4.1. If we denote (see, e.g., [17])

$$Re(t) = \frac{4d}{\gamma_0 c_1} \|z(t)\|_{L^4} \text{ (the Reynold number); } Ra(t) = \frac{4d^2 \|w(t)\|_{L^4}^2}{\gamma_0 k_0 c_1 \dot{c}_1} \text{ (the eigh number),}$$

.

Then, condition (4.2) is reduced to condition:

$$Re(t) + Ra(t) < 1$$

Remark 4.2. Theorem 4.1 is a generalization of the results of [17]. In [17], the boundary condition for the velocity of fluid is given by the standard boundary condition, that is, the homogeneous Dirichlet boundary condition, and the boundary condition for the temperature of fluid is given by the non-homogeneous Dirichlet boundary condition.

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